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A FUNCTIONAL CALCULUS BASED ON FEYNMAN-KAC FORMULA

BY

ANDRZEJ HULANICKI (WROCŁAW)

Abstract. It is proved that if

$$Hf = \int_0^\infty \lambda E(\lambda) f$$

is a spectral resolution of a Schrödinger operator $H = -\Delta + V$ on $\boldsymbol{R}^d$ with $V \in K^d_{\text{loc.}}$, $V(x) \geqslant 0$ and $V(x) \geqslant C|x|^\alpha$ for some $\alpha > 0$ and $|x| \geqslant C$, then there exists an N such that if $K \in C^N_c$, then the operator

$$\int_0^\infty K(\lambda) dE(\lambda)$$

is bounded on $L^p(\boldsymbol{R}^d)$, $1 \leqslant p < \infty$.

Let H be a self-adjoint (unbounded) operator on $L^2(\mathscr{M})$, where $\mathscr{M}$ is a measure space. We write its spectral resolution

$$Hf = \int_{-\infty}^{+\infty} \lambda\, dE(\lambda) f.$$

As we know, if $K \in L^\infty(R)$, then

$$E_K = \int_{-\infty}^{+\infty} K(\lambda)\, dE(\lambda)$$

is a bounded operator on $L^2(\mathscr{M})$ and

$$L^\infty(\mathscr{M}) \ni K \to E_K \in \mathscr{B}(L^2(\mathscr{M}))$$

is a $*$-homomorphism.

This is the simplest and the best known functional calculus.

QUESTION. Are there any reasonable conditions on K under which E_K is bounded on some $L^p(\mathscr{M})$, $p \neq 2$?

Of course in this generality the answer is "no".

In his book *Topics in Harmonic Analysis*... Stein [3] proved the following theorem, perhaps still the best one, specifying conditions on H under which the question has an answer.

Stein assumes that the operator H is the infinitesimal generator of a semi-group of operators $\{T_t\}_{t>0}$ such that

(2) $$\|T_t\|_{L^p, L^p} \leqslant 1 \quad \text{for all } 1 \leqslant p \leqslant \infty.$$

THEOREM (E. M. Stein). *Condition* (2) *and*

(3) $$K(\lambda) = \lambda \int_0^\infty e^{-\lambda\xi} m(\xi)\, d\xi \quad \textit{for some } m \in L^\infty(R^+),$$

imply that $\|E_k\|_{L^p, L^p} \leqslant C_p$ *for all* $1 < p < \infty$.

As we see, condition (3) implies that K is holomorphic in the right half-plane. However for some specific operators H the class of functions K on R^+ for which E_K is bounded on some L^p, $p \neq 2$, contains functions with compact support. This is the case of some Schrödinger operators.

These are operators of the form

$$H = -\frac{1}{2}\Delta + V(x),$$

where Δ is the laplacian on R^d and V is the potential, i.e. the operator of multiplication by the function V.

The following condition on V has been introduced by M. Aizenman and B. Simon in 1982 (cf. e.g. [1]):

(K_d^{loc}) $$\lim_{\alpha \to 0} \sup_{|x - x_0| < 1} \int_{|x-y| < \alpha} V(y)\varphi(x-y)\, dy = 0,$$

where

$$\varphi(x) = \begin{cases} |x|^{-d+2} & \text{if } d > 2, \\ \log|x| & \text{if } d = 2, \\ 1 & \text{if } d = 1. \end{cases}$$

THEOREM. *Assume that* V *satisfies* (K_d^{loc}), $V(x) \geqslant 0$, *and, for some* $\alpha > 0$, $V(x) \geqslant |x|^\alpha$ *for* $|x| > C'$. *Let*

$$N \geqslant \frac{d}{2(\alpha \wedge 2)} + 3.$$

Then, if $K \in C^N[0, \infty)$ *and*

(4) $$\sup\{e^{N\lambda}|K^{(j)}(\lambda)|: \lambda > 0\} < \infty, \quad j = 0, \ldots, N,$$

then $\|E_K\|_{L^1, L^1} < \infty$, *which, by interpolation, implies*

$$\|E_K\|_{L^p, L^p} < \infty \quad \textit{for all } 1 \leqslant p \leqslant \infty.$$

Remark. The class of functions defined by (4) is an algebra in which $C_c^N[0, \infty)$ is dense.

Proof. The proof is based on an old idea of Y. Katznelson (cf. e.g. [2]) which has been used many times by various authors.

Let $e(\xi) = e^{i\xi} - 1$. If $F \in C^1(-\pi, \pi)$ and $F(0) = 0$, then

$$F(\xi) = \sum \hat{F}(n)(e^{in\xi} - 1) + \sum \hat{F}(n) = \sum \hat{F}(n)\, e(n\xi).$$

Since, for a fixed n,

$$e(n\xi) = \sum_{k=1}^{\infty} \frac{(in)^k}{k!} \xi^k \quad \text{if } \|A\|_{L^1, L^1} < \infty,$$

we have $\|e(nA)\|_{L^1, L^1} < \infty$.

Suppose

(4) $$\|e(nA)\|_{L^1, L^1} \leqslant C|n|^M.$$

Then, of course, for $F \in C^{M+2}(-\pi, \pi)$ and $F(0) = 0$,

$$F(A) = \sum \hat{F}(n)\, e(nA) \in \mathscr{B}(L^1, L^1).$$

So, if $A = E_\varphi$, and the range of φ is contained in $(-\pi, \pi)$, then, by (1),

$$E_{F(\varphi)} = \int_{-\infty}^{+\infty} F(\varphi(\lambda)\, dE(\lambda) \in \mathscr{B}(L^1, L^1).$$

Now assume H is a Schrödinger operator which satisfies the assumption of the theorem. Then H is essentially self-adjoint, and non-negative. Let

$$Hf = \int_0^\infty \lambda\, dE(\lambda)$$

be its spectral resolution. We write

$$T_t f = \int_0^\infty e^{-\lambda t}\, dE(\lambda) f.$$

The Feynman-Kac formula says

$$T_t f(x) = \underset{x}{E} \exp\left[-\int_0^t v(b_s)\, ds\right] f(b_t),$$

where b is the Brownian motion in R^d. Hence, since $V(x) \geqslant 0$,

$$|T_t f(x)| \leqslant \underset{x}{E}\, |f(b_t)| = |f| * p_t, \quad \text{where} \quad p_t(x) = (2\pi t)^{-d/2} \exp\left[-\frac{\|x\|^2}{2t}\right].$$

Hence $\|T_t\|_{L^1, L^1} \leqslant 1$.

We put $T = T_1$ and estimate $\|e(nT) f\|_{L^1}$ in terms of $\|f\|_{L^1}$.

First we note that $e(nT) = AT$, where, by the spectral theorem,

$$\|A\|_{L^2, L^2} \leqslant \sup\{|\lambda^{-1}(e^{-i\lambda n} - 1)| : \lambda > 0\}.$$

We write

$$\|e(nT)f\|_{L^1} = \int |e(nT)f|dx = \int_{|x|\leqslant m} + \int_{|x|>m} = I_1 + I_2,$$

where $|x| = \max|x_i|$, $x = (x_1, \ldots, x_d)$. Then, by the Schwarz inequality,

$$I_1 \leqslant m^{d/2}\|e(nT)f\|_{L^2} \leqslant m^{d/2}\|A\|_{L^2,L^2}\|Tf\|_{L^2} \leqslant m^{d/2}|n|C_T\|f\|_{L^1}, \tag{5}$$

since, by M. Aizenman, B. Simon (cf. [1]), $V \in K_d^{\text{loc}}$, $V(x) \geqslant 0$ implies $\|Tf\|_{L^2} \leqslant C_T\|f\|_{L^1}$. On the other hand,

$$I_2 \leqslant \int_{|x|>m} \sum_{k=1}^{\infty} \frac{|n|^k}{|n|!} \mathop{E}_x \exp\left[-\int_0^k V(b_s)\,ds\right] |f(b_k)|dx.$$

Now we use the following well-known, and easy to prove fact (cf. [1]):

$$P_x\left\{\inf_{0\leqslant s\leqslant 1} |b_s| < \frac{1}{2}|x|\right\} \leqslant P_0\left\{\sup_{0\leqslant s\leqslant 1} |b_s| \geqslant \frac{1}{2}|x|\right\}$$

$$\leqslant 2dP_0\left\{\sup_{0\leqslant s\leqslant 1} b_s^1 \geqslant \frac{1}{2}|x|\right\} = 4dP_0\left\{b_1^1 \geqslant \frac{1}{2}|x|\right\} \leqslant Ce^{-\varepsilon|x|^2}$$

for some C and $\varepsilon > 0$ which depend only on d, and b^1 denotes the one-dimensional Brownian motion. Hence, for $|x| > C'$,

$$\mathop{E}_x \exp\left[-\int_0^k V(b_s)\,ds\right]|f(b_k)| \leqslant \mathop{E}_x \exp\left[-\int_0^1 V(b_s)\,ds\right]|f(b_k)|$$

$$\leqslant P_x\left\{\inf_{0\leqslant s\leqslant 1} |b_s| < \frac{1}{2}|x|\right\}\mathop{E}_x|f(b_k)| + \exp\left[-\frac{1}{2}|x|^\alpha \mathop{E}_x|f(b_k)|\right.$$

$$\leqslant (Ce^{-\varepsilon|x|^2} + e^{-|x|^\alpha/2})|f|*p_t(x).$$

Consequently,

$$\int_{|x|>m} \mathop{E}_x \exp\left[-\int_0^k V(b_s)\,ds\right]|f(b_k)| \leqslant c'e^{-\varepsilon' m^{\alpha\wedge 2}}\|f\|_{L^1}$$

for some c' and $\varepsilon' > 0$. Thus $I_2 \leqslant c'e^{|n|}e^{-\varepsilon' m^{\alpha\wedge 2}}\|f\|_{L^1}$.

Putting $m = c|n|^{1/(\alpha\wedge 2)}$ for sufficiently large c, by (5), we obtain

$$\|e(nT)\|_{L^1,L^1} \leqslant C|n|^{d/2(\alpha\wedge 2)+1}.$$

Thus for every $F \in C^N(-\pi, \pi)$ such that $F(0) = 0$ the function

$$K(\lambda) = F(e^{-\lambda}) \tag{6}$$

has the property $\|E_k\|_{L^1,L^1} < \infty$. It is easy to verify that functions of the form

(6) are precisely the ones which satisfy (4). This completes the proof of the theorem.

REFERENCES

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Institute of Mathematics
Wrocław University
pl. Grunwaldzki 2/4
50-384 Wrocław, Poland

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